Anomalous criticality near semimetal-to-superfluid quantum phase transition in a two-dimensional Dirac cone model

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We analyze the scaling behavior at and near a quantum critical point separating a semimetallic from a superfluid phase. To this end we compute the renormalization group flow for a model of attractively interacting electrons with a linear dispersion around a single Dirac point. We study both ground state and finite temperature properties. In two dimensions, the electrons and the order parameter fluctuations exhibit power-law scaling with anomalous scaling dimensions. The quasi-particle weight and the Fermi velocity vanish at the quantum critical point. The order parameter correlation length turns out to be infinite everywhere in the semimetallic ground state.

1 Introduction

Quantum phase transitions in interacting electron systems are traditionally described by an effective order parameter theory, which was pioneered by Hertz [1] and Millis [2]. In that approach, an order parameter field is introduced via a Hubbard-Stratonovich transformation and the electrons are subsequently integrated out. The resulting effective action for the order parameter is then truncated at quartic order and analyzed by standard scaling and renormalization group (RG) techniques.

However, more recent studies revealed that the Hertz-Millis approach is often not applicable, especially in low dimensional systems [3, 4]. For electron systems with a Fermi surface the electronic excitation spectrum is gapless. As a consequence, integrating out the electrons may lead to singular interactions in the effective order parameter action, which cannot be approximated by a local quartic term. Therefore it is better to keep the electronic degrees of freedom in the theory, treating them on equal footing with the bosonic order parameter field. Several coupled boson-fermion systems exhibiting quantum criticality have been analyzed in the last decade by various methods [5–10].

Recently, a Dirac cone model describing attractively interacting electrons with a linear energy-momentum dispersion was introduced to model a continuous quantum phase transition from a semimetal to a superfluid [11]. The scaling behavior at the quantum critical point (QCP) was studied by coupled boson-fermion flow equations derived within the functional RG framework. It was shown that electrons and bosons acquire anomalous scaling dimensions in dimensions $d < 3$, implying non-Fermi liquid behavior and non-Gaussian order parameter fluctuations.

In this work we extend the analysis of the Dirac cone model in various directions, with a focus on the two-dimensional case. First, we allow for a renormalization of the Fermi velocity of the electrons, which was omitted in Ref. [11], but indeed turns out to be important. Second, we study the behavior upon approaching the QCP from the semimetallic phase at zero temperature. While the pairing susceptibility exhibits the expected power-law scaling, we find that the correlation length is infinite everywhere in the

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semimetallic phase. Finally, we compute the scaling behavior of the susceptibility and the correlation length in the finite temperature quantum critical regime.

The paper is structured as follows. In Sec. 2 we define the Dirac cone model and the corresponding action. The derivation of the flow equations is described in Sec. 3, and results are presented in Sec. 4. We conclude with a short summary in Sec. 5.

2   Dirac cone model

We consider a model of electrons with a linear dispersion relation $\epsilon_{\mathbf{k}\alpha} = \alpha v_f |\mathbf{k}|$, with $\alpha = \pm 1$, corresponding to two "Dirac cones" with positive ($\alpha = 1$) and negative ($\alpha = -1$) energy. The chemical potential is chosen as $\mu = 0$, such that in the absence of interactions states with negative energy are filled, while states with positive energy are empty. The Fermi surface thus consists of only one point, the "Dirac point" at $\mathbf{k} = \mathbf{0}$, where the two Dirac cones touch. The action of the interacting system with a local attractive interaction $U < 0$ is given by [11]

$$
\begin{aligned}
\mathcal{S}[\psi, \bar{\psi}] &= \int_{k\alpha\sigma} \bar{\psi}_{k\alpha\sigma}(-ik_0 + \epsilon_{\mathbf{k}\alpha})\,\psi_{k\alpha\sigma} + U \int_{k\alpha} \int_{k'\alpha'} \int_q \bar{\psi}_{-k,\alpha\downarrow}\bar{\psi}_{k+q,\alpha\uparrow}\psi_{k'+q,\alpha'\uparrow}\psi_{-k',\alpha'\downarrow} \\
&\quad + \int_{k\alpha\sigma} m_\alpha\,\bar{\psi}_{k\alpha\sigma}\psi_{k\alpha\sigma}\,,
\end{aligned}
\tag{1}
$$

where ψ and $\bar{\psi}$ are fermionic fields. The variables $k = (k_0, \mathbf{k})$ and $q = (q_0, \mathbf{q})$ collect Matsubara frequencies and momenta, and we use the short-hand notation $\int_k = T \sum_{k_0} \int \frac{d^d\mathbf{k}}{(2\pi)^d}$ for momentum integrals and frequency sums; $\int_{k\alpha}$ includes also the sum over the band index α and $\int_{k\alpha\sigma}$ includes in addition the spin sum over $\sigma = \uparrow, \downarrow$. Momentum integrations are restricted by the ultraviolet cutoff $v_f |\mathbf{k}| < \Lambda_0$.

In Ref. [11] it was tacitly assumed that the interaction does not shift the upper and lower Dirac cone with respect to each other. To compensate for self-energy contributions which in fact do generate such a shift, we have added a fermionic mass term with a U-dependent mass m_α to the action $\mathcal{S}[\psi, \bar{\psi}]$. This term is tuned such that the Dirac cones touch each other at $\mathbf{k} = 0$ for any U.

The kinetic energy in Eq. (1) is a toy version of the dispersion for electrons moving on a honeycomb lattice as in graphene, where the momentum dependence is entangled with a pseudospin degree of freedom related to the two-atom structure of the unit cell [12]. Note that the kinetic energy and the interaction in Eq. (1) are both diagonal in the spin indices. By contrast, in Dirac fermion models describing surface states of certain three-dimensional topological insulators the spin orientation is correlated with the momentum [13, 14]. We are not aware of a physical realization of the model Eq. (1) in a real material. The model was designed to analyze the quantum phase transition between a semimetal and a superfluid in the simplest possible setting. Although the model (1) is reminiscent of the Gross-Neveu model [15], it is not equivalent to it. In particular, for the Gross-Neveu model there is no choice of a spinor basis in which the kinetic and potential energies are both spin-diagonal.

The attractive interaction favors spin singlet pairing [11]. Therefore, we decouple the interaction in the s-wave spin-singlet pairing channel by introducing a complex bosonic Hubbard-Stratonovich field ϕ conjugate to the bilinear composite of fermionic fields $U \int_{k\alpha} \psi_{k+q,\alpha\uparrow}\psi_{-k,\alpha\downarrow}$. This yields a functional integral over ψ, $\bar{\psi}$ and ϕ with the fermion-boson action

$$
\begin{aligned}
\mathcal{S}[\psi, \bar{\psi}, \phi] &= \int_{k\alpha\sigma} \bar{\psi}_{k\alpha\sigma}(-ik_0 + \epsilon_{\mathbf{k}\alpha} + m_\alpha)\,\psi_{k\alpha\sigma} - \int_q \phi_q^* \frac{1}{U}\,\phi_q \\
&\quad + \int_{k\alpha} \int_q \left(\bar{\psi}_{-k,\alpha\downarrow}\bar{\psi}_{k+q,\alpha\uparrow}\,\phi_q + \psi_{k+q,\alpha\uparrow}\psi_{-k,\alpha\downarrow}\,\phi_q^* \right),
\end{aligned}
\tag{2}
$$

where ϕ^* is the complex conjugate of ϕ. The boson mass $\delta = -1/U > 0$ is the control parameter for the quantum phase transition. In mean-field theory a continuous transition between the semimetallic and

a superfluid phase occurs at the quantum critical point $U_{qc}^{\mathrm{MF}} = -2\pi v_f^2/\Lambda_0$ in two dimensions [11]. For technical reasons explained in Sec. 3, we will supplement the bosonic part of the action by adding a term of the form $\int_q \phi_q^*(Z_b q_0^2 + A_b \mathbf{q}^2)\phi_q$, which regularizes the flow at high scales without influencing the low-energy properties of the system. The extra term corresponds to a replacement of the local interaction U by a q-dependent interaction $U(q) = U/[1 - U(Z_b q_0^2 + A_b \mathbf{q}^2)]$, which decreases at large momenta and frequencies. From now on we set $v_f = 1$.

3 Renormalization group

Our aim is to derive scaling properties of the electrons and the order parameter fluctuations near the quantum phase transition. To this end we derive flow equations for the scale-dependent effective action $\Gamma^\Lambda[\psi, \bar{\psi}, \phi]$ within the functional RG framework for fermionic and bosonic degrees of freedom [16–19]. Starting from the bare fermion-boson action $\Gamma^{\Lambda=\Lambda_0}[\psi, \bar{\psi}, \phi] = \mathcal{S}[\psi, \bar{\psi}, \phi]$ in Eq. (2), fermionic and bosonic fluctuations are integrated simultaneously, proceeding from higher to lower scales as parametrized by the continuous flow parameter Λ. In the infrared limit $\Lambda \to 0$, the fully renormalized effective action $\Gamma[\psi, \bar{\psi}, \phi]$ is obtained. The flow of Γ^Λ is governed by the exact functional flow equation [16]

$$\frac{d}{d\Lambda}\Gamma^\Lambda[\psi, \bar{\psi}, \phi] = \mathrm{Str}\, \frac{\partial_\Lambda R^\Lambda}{\Gamma^{(2)\Lambda}[\psi, \bar{\psi}, \phi] + R^\Lambda}\ , \tag{3}$$

where $\Gamma^{(2)\Lambda}$ denotes the second functional derivative with respect to the fields and R^Λ is the infrared regulator (to be specified below). The supertrace (Str) traces over all indices, with an additional minus sign for fermionic contractions.

3.1 Truncation

The functional flow equation Eq. (3) cannot be solved exactly. We therefore truncate the effective action with the objective to capture the essential renormalization effects. Our ansatz for Γ^Λ is a slight generalization of the truncation used in Ref. [11] of the following form

$$\Gamma^\Lambda = \Gamma_{\bar{\psi}\psi}^\Lambda + \Gamma_{\phi^*\phi}^\Lambda + \Gamma_{|\phi|^4}^\Lambda + \Gamma_{\psi^2\phi^*}^\Lambda\ , \tag{4}$$

where

$$\Gamma_{\bar{\psi}\psi}^\Lambda = \int_{k\alpha\sigma} \bar{\psi}_{k\alpha\sigma}(-iZ_f^\Lambda k_0 + A_f^\Lambda \epsilon_{\mathbf{k}\alpha} + m_\alpha^\Lambda)\,\psi_{k\alpha\sigma}\ , \tag{5}$$

$$\Gamma_{\phi^*\phi}^\Lambda = \int_q \phi_q^*\left(Z_b^\Lambda q_0^2 + A_b^\Lambda \mathbf{q}^2 + \delta^\Lambda\right)\phi_q\ , \tag{6}$$

$$\Gamma_{|\phi|^4}^\Lambda = \frac{u^\Lambda}{8}\int_{q,q',p} \phi_{q+p}^*\phi_{q'-p}^*\phi_{q'}\phi_q\ , \tag{7}$$

$$\Gamma_{\psi^2\phi^*}^\Lambda = g^\Lambda \int_{k\alpha}\int_q \left(\bar{\psi}_{-k,\alpha\downarrow}\bar{\psi}_{k+q,\alpha\uparrow}\,\phi_q + \psi_{k+q,\alpha\uparrow}\psi_{-k,\alpha\downarrow}\,\phi_q^*\right)\ . \tag{8}$$

The momentum and frequency dependence of $\Gamma_{\phi^*\phi}^\Lambda$, and also the bosonic interaction $\Gamma_{|\phi|^4}^\Lambda$, are generated by fermionic fluctuations. The fermion-boson vertex $\Gamma_{\psi^2\phi^*}^\Lambda$ is actually not renormalized within our truncation. The usual one-loop vertex correction, which is formally of order g^3, vanishes in the normal phase due to particle conservation [19]. Hence, the coupling g remains invariant at its bare value $g = 1$ in the course of the flow.

In Ref. [11] a restricted version of the ansatz Eq. (4) with $A_f^\Lambda = Z_f^\Lambda$ and $A_b^\Lambda = Z_b^\Lambda$ was used, since it was assumed that frequency and momentum dependences renormalize similarly. However, a closer inspection reveals that this is not the case. In particular, it turns out that one-loop contributions to the flow of A_f^Λ

cancel, while Z_f^Λ flows to infinity at the QCP. This asymmetry between momentum and frequency scaling generates also a significant difference between A_b^Λ and Z_b^Λ.

The initial conditions for the fermionic renormalization factors are $Z_f^{\Lambda_0} = A_f^{\Lambda_0} = 1$. The initial condition for the bosonic mass is $\delta^{\Lambda_0} = -1/U$, and the quartic bosonic interaction u is initially zero. The initial conditions for Z_b and A_b corresponding to the bare action in Eq. (2) are $Z_b^{\Lambda_0} = A_b^{\Lambda_0} = 0$. However, starting the flow with $Z_b^{\Lambda_0} = A_b^{\Lambda_0} = 0$ leads to very large transient anomalous dimensions at the initial stage of the flow (for Λ near Λ_0), which complicates the analysis in a (high energy) regime which is physically not interesting. The qualitative behavior of the low energy flow ($\Lambda \ll \Lambda_0$) and the critical exponents do not depend on the initial values of Z_b and A_b. We therefore add a term $\int_q \phi_q^* (q_0^2 + \mathbf{q}^2) \phi_q$ to the bare action, corresponding to initial values $Z_b^{\Lambda_0} = A_b^{\Lambda_0} = 1$. This term regularizes the model by suppressing the interaction for large momentum and energy transfers.

As regulators in the flow equation (3) we choose momentum dependent Litim functions [20], supplemented by a mass shift for the fermions,

$$
\begin{aligned}
R_{f\alpha}^\Lambda(\mathbf{k}) &= A_f \left[-\Lambda \, \mathrm{sgn}(\epsilon_{\mathbf{k}\alpha}) + \epsilon_{\mathbf{k}\alpha} \right] \theta(\Lambda - |\epsilon_{\mathbf{k}\alpha}|) + \delta m_\alpha^\Lambda , \\
R_b^\Lambda(\mathbf{q}) &= A_b \left(-\Lambda^2 + \mathbf{q}^2 \right) \theta \left(\Lambda^2 - \mathbf{q}^2 \right) ,
\end{aligned}
\tag{9}
$$

where δm_α^Λ is chosen such that it cancels m_α^Λ in Eq. (5) at each scale Λ. Note that we have set $v_f = 1$, such that Λ is a common momentum cutoff for fermions and bosons. Adding the regulator functions to the quadratic terms in the effective action Γ^Λ yields the inverse of the regularized propagators, which thus have the form

$$
G_{f\alpha}^\Lambda(k) = \left[iZ_f^\Lambda k_0 - A_f^\Lambda \epsilon_{\mathbf{k}\alpha} - m_\alpha^\Lambda + R_{f\alpha}^\Lambda(\mathbf{k}) \right]^{-1} = \frac{1}{iZ_f^\Lambda k_0 - A_f^\Lambda \mathrm{sgn}(\epsilon_{\mathbf{k}\alpha}) \max(\Lambda, |\epsilon_{\mathbf{k}\alpha}|)} , \tag{10}
$$

$$
G_b^\Lambda(q) = \left[-Z_b^\Lambda q_0^2 - A_b^\Lambda \mathbf{q}^2 - \delta^\Lambda + R_b^\Lambda(\mathbf{q}) \right]^{-1} = -\frac{1}{Z_b^\Lambda q_0^2 + A_b^\Lambda \max(\Lambda^2, \mathbf{q}^2) + \delta^\Lambda} . \tag{11}
$$

Symmetry breaking in interacting Fermi systems is often studied by extending the model to an arbitrary number of fermion flavors N_f, and expanding in the parameter $1/N_f$. Our truncation captures the leading contributions for large N_f. The low energy behavior is captured correctly also to leading order in ϵ, where $\epsilon = 3 - d$ is the deviation from the critical spatial dimension $d_c = 3$, below which anomalous scaling sets in.

3.2   Flow equations

The flow equations are obtained by inserting the ansatz Eq. (4) for Γ^Λ into the exact functional flow equation Eq. (3) and comparing coefficients. For a concise formulation, we use the following short-hand notation for a cutoff derivative and loop integration:

$$
\int_k' = T \sum_{k_0} \int \frac{d^d k}{(2\pi)^d} \sum_{s=b,f} \left(-\partial_\Lambda R_s^\Lambda \right) \partial_{R_s^\Lambda} . \tag{12}
$$

The scale-derivatives of the regulators read

$$
\begin{aligned}
\partial_\Lambda R_{f\alpha}^\Lambda(\mathbf{k}) &= -A_f^\Lambda \, \mathrm{sgn}(\epsilon_{\mathbf{k}\alpha}) \, \theta(\Lambda - |\epsilon_{\mathbf{k}\alpha}|) , \\
\partial_\Lambda R_b^\Lambda(\mathbf{q}) &= -2A_b^\Lambda \Lambda \, \theta \left(\Lambda^2 - \mathbf{q}^2 \right) ,
\end{aligned}
\tag{13}
$$

where terms proportional to $\partial_\Lambda A_f^\Lambda$ and $\partial_\Lambda A_b^\Lambda$ are neglected (as usual, see [16]). The contribution from the mass shift, $\partial_\Lambda \delta m_\alpha^\Lambda$, is also discarded. It is formally of higher order (in a loop expansion) than the terms kept, and it does not affect the qualitative behavior. Note that the cutoff derivative in Eq. (12) acts only on the explicit cutoff dependence introduced via the regulator functions.

We thus obtain the following equations for the flow of parameters in our ansatz for Γ^Λ:

$$\partial_\Lambda Z_f^\Lambda = (g^\Lambda)^2 \int_q' \frac{\partial}{i\partial k_0} G_{f\alpha}^\Lambda(q-k) \, G_b^\Lambda(q) \bigg|_{k=0} , \tag{14}$$

$$\partial_\Lambda A_f^\Lambda = 0 , \tag{15}$$

$$\partial_\Lambda \delta^\Lambda = (g^\Lambda)^2 \int_{k\alpha}' G_{f\alpha}^\Lambda(k) \, G_{f\alpha}^\Lambda(-k) + \frac{u^\Lambda}{2} \int_q' G_b^\Lambda(q) , \tag{16}$$

$$\partial_\Lambda Z_b^\Lambda = \frac{1}{2}\frac{\partial^2}{\partial q_0^2} \, (g^\Lambda)^2 \int_{k\alpha}' G_{f\alpha}^\Lambda(k+q) \, G_{f\alpha}^\Lambda(-k) \bigg|_{q=0} , \tag{17}$$

$$\partial_\Lambda A_b^\Lambda = \frac{1}{2}\frac{\partial^2}{\partial q_1^2} \, (g^\Lambda)^2 \int_{k\alpha}' G_{f\alpha}^\Lambda(k+q) \, G_{f\alpha}^\Lambda(-k) \bigg|_{q=0} , \tag{18}$$

$$\partial_\Lambda u^\Lambda = -4(g^\Lambda)^4 \int_{k\alpha}' \left[G_{f\alpha}^\Lambda(-k)\right]^2 \left[G_{f\alpha}^\Lambda(k)\right]^2 + \frac{5}{4}(u^\Lambda)^2 \int_q' \left[G_b^\Lambda(q)\right]^2 , \tag{19}$$

$$\partial_\Lambda g^\Lambda = 0 . \tag{20}$$

The flow equations for Z_f, δ, Z_b, u, and g are the same as in Ref. [11]. The momentum derivative in the flow equation for A_b is with respect to the first (or any other) component of $\mathbf{q}$. All frequency sums and momentum integrations in the above flow equations can be performed analytically, both at zero and finite temperature.

Explicit Λ-dependences in the flow equations can be absorbed by using rescaled variables

$$\tilde{\delta}^\Lambda = \frac{\delta^\Lambda}{\Lambda^2 A_b^\Lambda} , \quad \tilde{g}^\Lambda = \frac{g^\Lambda}{\Lambda^{\frac{3-d}{2}}\sqrt{Z_f^\Lambda A_f^\Lambda A_b^\Lambda}} , \quad \tilde{u}^\Lambda = \frac{u^\Lambda}{\Lambda^{3-d}\sqrt{Z_b^\Lambda (A_b^\Lambda)^3}} . \tag{21}$$

At $T > 0$ one also has to use rescaled temperatures $\tilde{T}_b^\Lambda = T\frac{\sqrt{Z_b^\Lambda/A_b^\Lambda}}{\Lambda}$ and $\tilde{T}_f^\Lambda = T\frac{Z_f^\Lambda}{A_f^\Lambda \Lambda}$ to absorb Λ. Anomalous dimensions are defined as usual by logarithmic derivatives of the renormalization factors

$$\eta_b^A = -\frac{d\log A_b^\Lambda}{d\log\Lambda} , \quad \eta_b^Z = -\frac{d\log Z_b^\Lambda}{d\log\Lambda} , \quad \eta_f^A = -\frac{d\log A_f^\Lambda}{d\log\Lambda} , \quad \eta_f^Z = -\frac{d\log Z_f^\Lambda}{d\log\Lambda} . \tag{22}$$

Note that $\eta_f^A = 0$, since A_f^Λ does not flow.

4 Results

We now discuss the scaling behavior as obtained from a solution of the flow equations, focussing mostly on the two-dimensional case. Anomalous scaling dimensions occur in dimensions $d < 3$ [11]. We first discuss the ground state, including the quantum critical point, and then finite temperatures. Numerical results depending on the ultraviolet cutoff Λ_0 will be presented for the choice $\Lambda_0 = 1$.

4.1 Quantum critical point

To reach the quantum critical point one has to tune the bare interacting to a special value U_{qc} such that the bosonic mass δ^Λ scales to zero for $\Lambda \to 0$. In two dimensions we find $U_{qc} \approx -15.646$ for $\Lambda_0 = 1$, which is about a factor 2.5 larger than the mean-field value. For $U = U_{qc}$ the rescaled variables defined in Eq. (21) scale to a non-Gaussian fixed point, with finite anomalous dimensions in any dimension $d < 3$. Since g^Λ does not flow at all, the scale invariance of $\tilde{g}^\Lambda$ at the fixed point leads to a simple relation between the anomalous dimensions,

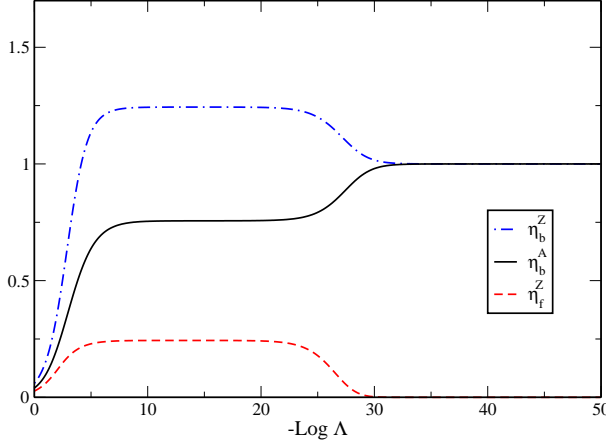$$\eta_b^A + \eta_f^Z + \eta_f^A = 3 - d . \tag{23}$$

Fig. 1 Flow of anomalous dimensions as a function of Λ in the semimetallic ground state for a choice of U close to the QCP.

Furthermore, since the flow of Z_b and A_b is determined entirely by a convolution of two fermionic propagators, the differences of anomalous dimensions for frequency and momentum scaling of fermions and bosons are linked by a simple condition, which can be expressed as

$$\eta_b^Z - \eta_b^A = 2(\eta_f^Z - \eta_f^A) \,. \tag{24}$$

Due to $\eta_f^A = 0$ the above relations reduce to $\eta_b^A + \eta_f^Z = 3 - d$ and $\eta_b^Z - \eta_b^A = 2\eta_f^Z$. Solving the fixed point equations we obtain the numerical values $\eta_b^A \approx 0.75$, $\eta_b^Z \approx 1.25$, and $\eta_f^Z \approx 0.25$ in two dimensions. Hence, at the quantum critical point the order parameter exhibits non-Gaussian critical fluctuations with different anomalous scaling dimensions for momentum and frequency dependences. Furthermore, the fermionic quasiparticle weight ($\propto Z_f^{-1}$) vanishes, which implies non-Fermi liquid behavior. Since A_f remains finite, the Fermi velocity also vanishes at the quantum critical point. This last point was missed in Ref. [11]. Due to the different anomalous dimensions for momentum and frequency scaling, the dynamical exponent z acquires an anomalous dimension, too. In the bare action $\mathcal{S}$ one has $z_0 = 1$ for bosons and fermions. At the quantum critical point, we find

$$z_f = 1 + \eta_f^Z - \eta_f^A = z_b = 1 + \frac{\eta_b^Z - \eta_b^A}{2} \approx 1.25 \,. \tag{25}$$

The equality between z_b and z_f follows from Eq. (24).

4.2 Semimetallic ground state

For $|U| < |U_{qc}|$, the bosonic mass δ^Λ saturates at a finite value for $\Lambda \to 0$, corresponding to a finite pairing susceptibility $\chi = \lim_{\Lambda \to 0} (\delta^\Lambda)^{-1}$. The fermionic Z-factor also saturates, such that $\eta_f^Z \to 0$. Hence, fermionic quasiparticles survive in the semimetallic state. However, A_b^Λ and Z_b^Λ do not saturate, but rather diverge as Λ^{-1}, such that $\eta_b^A, \eta_b^Z \to 1$. This is illustrated in Fig. 1, where the anomalous dimensions are plotted as a function of Λ for a choice of U close to the QCP. The QCP scaling is seen at intermediate scales, before the anomalous dimensions saturate at the asymptotic values $\eta_f^Z = 0$ and $\eta_b^A = \eta_b^Z = 1$ for $\Lambda \to 0$. A finite anomalous dimension away from the critical point is surprising at first sight. However, it can be explained quite easily. An explicit calculation shows that the leading small momentum and small frequency dependence of the fermionic particle-particle bubble is *linear* in two dimensions, as long as the propagators have a finite quasi-particle weight. In presence of an infrared cutoff this linear behavior is replaced by a quadratic behavior (as in our ansatz), but the prefactors of the quadratic terms diverge linearly in the limit $\Lambda \to 0$, reflecting thus the true asymptotic behavior.
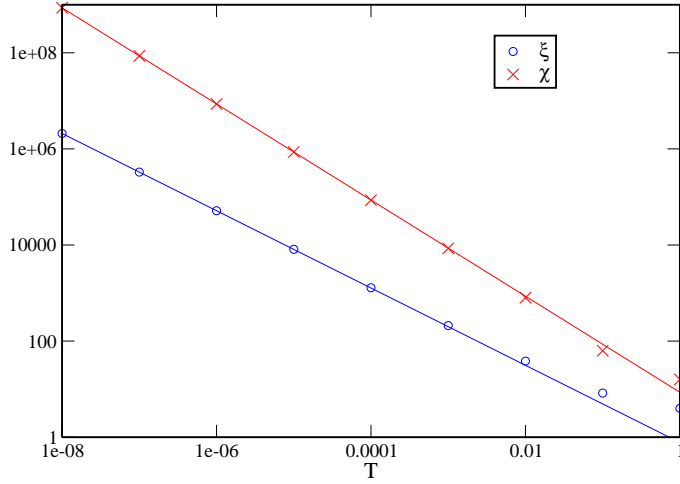
Fig. 2 Temperature dependence of the pairing susceptibility χ and the correlation length ξ in a double-logarithmic plot for $U = U_{qc}$. At low temperatures the calculated points lie on straight lines, corresponding to power-law behavior.

The divergences of A_b^Λ and Z_b^Λ imply that the correlation length and correlation time of pairing fluctuations are always infinite in the semimetallic ground state, not only at the QCP. This is consistent with the observation that the linear momentum and frequency dependence of the particle-particle bubble leads to a power-law decay of its Fourier transform at long space or time distances, instead of the usual exponential decay. The divergent correlation length suggests that the entire semimetallic phase is in some sense "quantum critical". This point of view has indeed been adopted in theories of interaction effects in graphene, where the particle-hole symmetric (Dirac) point is interpreted as a QCP separating the electron-doped from the hole-doped Fermi liquid. Scaling concepts could then be used to compute thermodynamic [21] and transport [22] properties of graphene near the Dirac point.

The pairing susceptibility χ is generically finite in the semimetallic ground state and diverges upon approaching the QCP. From a numerical solution of the flow equations in two dimensions we have obtained the power-law

$$\chi(U) \propto (|U_{qc}| - |U|)^{-\gamma_0} , \quad \text{with } \gamma_0 \approx 1.725 . \tag{26}$$

4.3 Temperature scaling

We now present results for $U = U_{qc}$ and $T > 0$, that is, we approach the QCP as a function of temperature. At finite temperature the fermionic propagator is cut off by temperature itself, since fermionic Matsubara frequencies are bounded by $|k_0| \geq \pi T$, and the bosonic propagator is regularized by the finite bosonic mass δ. Hence, the flow of all unscaled variables saturates for $\Lambda \to 0$. Power-laws are obtained for these saturated variables as a function of temperature. In particular,

$$A_b \propto T^{-\bar{\eta}_b^A} , \quad Z_b \propto T^{-\bar{\eta}_b^Z} , \quad Z_f \propto T^{-\bar{\eta}_f^Z} , \tag{27}$$

with $\bar{\eta}_b^A \approx 0.60$, $\bar{\eta}_b^Z \approx 1.00$, and $\bar{\eta}_f^Z \approx 0.20$ in two dimensions.

In Fig. 2 we show the temperature dependence of the susceptibility χ and the correlation length ξ, as obtained from a numerical solution of the flow equations at various temperatures in two dimensions. The susceptibility is given by the inverse bosonic mass δ at the end of the flow ($\Lambda \to 0$), the correlation length by $\xi = \sqrt{Z_b/\delta}$. Both quantities obey power-laws at low temperatures, namely

$$\chi(T) \propto T^{-\gamma} , \quad \text{with } \gamma \approx 1.00 , \tag{28}$$

$$\xi(T) \propto T^{-\nu} , \quad \text{with } \nu \approx 0.80 . \tag{29}$$

Note that we use the letters γ and ν for the exponents by applying the classical definition near a thermal phase transition $\chi \propto (T - T_c)^{-\gamma}$ and $\xi \propto (T - T_c)^{-\nu}$ to the present situation where $T_c = 0$. The correlation length exponent obeys $\nu = z_b^{-1}$, which corresponds to a T^{-1} scaling of the correlation time ξ_τ in accordance with general scaling arguments for quantum phase transitions. The exponents γ and ν obey the classical scaling relation $\gamma = (2 - \eta_b^A)\nu$.

5   Conclusion

We have analyzed the critical properties near a quantum phase transition between a semimetallic and a superfluid phase in a two-dimensional model of attractively interacting electrons with a Dirac cone dispersion, correcting and extending a previous work [11]. We have studied coupled flow equations for the fermionic degrees of freedom and the bosonic fluctuations associated with the superfluid order parameter. Both fermions and bosons acquire anomalous scaling dimensions at the QCP, corresponding to non-Fermi liquid behavior and non-Gaussian pairing fluctuations. Allowing for distinct renormalization factors for momentum and frequency scaling, we have found that they differ substantially at the QCP. In particular, the Fermi velocity vanishes. We have also analyzed the semimetallic ground state away from the QCP in more detail than previously, finding that the correlation length for pairing fluctuations is always infinite, not only at the QCP. Finally, we have studied the scaling behavior upon approaching the QCP as a function of temperature. The susceptibility and the correlation length obey power-laws in temperature, as expected, and the corresponding critical exponents obey the classical scaling relation.

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